AJAE Appendix:
'On the (Mis)Use of Wealth as a Proxy for Risk Aversion'

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Appendix

General method for deriving $\frac{\partial w_q}{\partial z_a}$ *and* $\frac{\partial w_q}{\partial z_p}$

Following Caputo (2005), the principal-agent problem can be formulated as equivalent to the maximization of an isoperimetric Hamiltonian H defined by

(A1) $$H \equiv V[z_p + q - w(q, z_a, z_p)] f(q \mid e) + U[z_a + w(q, z_a, z_p)]\{\lambda f(q \mid e) + \mu f_e(q \mid e)\}$$

Solving by backward induction, we first consider comparative statics over the efficient contract given a fixed effort level. The first- and second-order conditions for w are respectively

(A2) $$\frac{\partial H}{\partial w} = -V' f + \lambda f U' + \mu U' f_e = 0 \text{, and}$$

(A3) $$\frac{\partial^2 H}{\partial w^2} = V'' f + \lambda f U'' + \mu U'' f_e < 0 .$$

The first step in the comparative statics exercise consists in applying the univariate Implicit Function Theorem so as to obtain $\frac{\partial w}{\partial \lambda}$, $\frac{\partial w}{\partial \mu}$, $\frac{\partial w}{\partial z_a}$, and $\frac{\partial w}{\partial z_p}$. This yields

(A4) $$\frac{\partial w}{\partial \lambda} = -\frac{U' f}{U''[\lambda f + \mu f_e] + V'' f} = \frac{-U' f}{\left(\frac{\partial^2 H}{\partial w^2}\right)} > 0$$

(A5) $$\frac{\partial w}{\partial \mu} = -\frac{U' f_e}{U''[\lambda f + \mu f_e] + V'' f} = \frac{-U' f_e}{\left(\frac{\partial^2 H}{\partial w^2}\right)} ,$$

(A6) $$\frac{\partial w}{\partial z_a} = \frac{V'' f}{U''[\lambda f + \mu f_e] + V'' f} - 1 = \frac{V'' f}{\left(\frac{\partial^2 H}{\partial w^2}\right)} > -1 \text{, and}$$

(A7) $$\frac{\partial w}{\partial z_p} = \frac{V'' f}{U''[\lambda f + \mu f_e] + V'' f} = \frac{V'' f}{\left(\frac{\partial^2 H}{\partial w^2}\right)} > 0 .$$

Given that both the monotone likelihood ratio property and the convexity of the distribution function condition hold, f_e is negative (positive) for low (high) realizations of q, and so $\frac{\partial w}{\partial \mu}$ is indeterminate.

The second step consists in applying the multivariate Implicit Function Theorem on the system defined by the individual rationality (IR) and the (first-order) incentive compatibility (IC') constraints, taking $w(q, \lambda, \mu, z_a, z_p)$ to be implicitly defined from the first-order condition of the Hamiltonian. The system is such that

(A8) $$IR \equiv \int_{\underline{q}}^{\bar{q}} U[z_a + w(q, \lambda, \mu, z_a, z_p)] f(q \mid e) dq - \psi(e) - U(z_a) = 0 \text{, and}$$

(A9) $$IC \equiv \int_{\underline{q}}^{\bar{q}} U[z_a + w(q, \lambda, \mu, z_a, z_p)] f_e(q \mid e) - \psi'(e) = 0 .$$

Taking partial derivatives of equations A8 and A9 yields:

(A10) $$IR_\lambda = \int_{\underline{q}}^{\bar{q}} U' \frac{\partial w}{\partial \lambda} f dq ,$$

(A11) $$IR_\mu = \int_{\underline{q}}^{\bar{q}} U' \frac{\partial w}{\partial \mu} f dq$$

(A12) $$IC_\lambda = \int_{\underline{q}}^{\bar{q}} U' \frac{\partial w}{\partial \lambda} f_e dq$$

(A13) $IC_\mu = \int_{\underline{q}}^{\bar{q}} U' \frac{\partial w}{\partial \mu} f_e dq$

(A14) $IR_a = \int_{\underline{q}}^{\bar{q}} U' \left(1 + \frac{\partial w}{\partial z_a}\right) fdq - U'$

(A15) $IR_p = \int_{\underline{q}}^{\bar{q}} U' \frac{\partial w}{\partial z_p} fdq$, and

(A16) $IC_a = \int_{\underline{q}}^{\bar{q}} U' \left(1 + \frac{\partial w}{\partial z_a}\right) f_e dq$.

Note that $IR_\lambda > 0$, $IC_\mu > 0$, and $IR_p > 0$ whereas IR_μ, IC_λ, IR_a, IC_a, and IC_p are not signable due to the fact that f_e necessarily takes on positive and negative values at different values of q, which follows from the fact that $\int_{\underline{q}}^{\bar{q}} f_e(q \mid e) dq = 0$.

The following identities, however, can be established from the above equations: $IC_p = IC_a$, $IR_a = IR_p$, and $IR_\mu = IC_\lambda$. For $i \in \{a, p\}$, an application of the multivariate Implicit Function Theorem to the system defined by $IR[z_a, z_p, \lambda, \mu, w(\cdot, z_a, z_p, \lambda, \mu)] = 0$ and $IC[z_a, z_p, \lambda, \mu, w(\cdot, z_a, z_p, \lambda, \mu)] = 0$ yields

(A17) $\frac{\partial \mu}{\partial z_i} = \frac{IR_i IC_\lambda - IR_\lambda IC_i}{IC_\mu IR_\lambda - IC_\lambda IR_\mu}$, and

(A18) $\frac{\partial \lambda}{\partial z_i} = \frac{IR_i IC_\mu - IR_\mu IC_i}{IC_\mu IR_\lambda - IC_\lambda IR_\mu}$.

Due to the indeterminacy of the components these partial derivatives are not in general signable, and an application of the Cauchy-Schwarz inequality to the identical denominator of the above partial derivatives reveals that it is positive, but even then the only theoretical implications regarding the sign of the partial derivatives in equations A17 and A18 is that $\operatorname{sgn}\left[\frac{\partial \mu}{\partial z_i}\right] = \operatorname{sgn}\left[\frac{IC_\lambda}{IR_\lambda} - \frac{IC_i}{IR_i}\right]$ and $\operatorname{sgn}\left[\frac{\partial \lambda}{\partial z_i}\right] = \operatorname{sgn}\left[\frac{IR_i}{IC_i} - \frac{IR_\mu}{IC_\mu}\right]$. In particular, we need to sign $\frac{\partial \mu}{\partial z_i}$ in order to sign $\frac{\partial w_q}{\partial z_i}$. Again, the indeterminacy of the derivatives stems from the fact that f_e is neither identically zero, non-positive, or non-negative.

If we subsequently try to account for the fact that the second-best optimal effort level also changes in principle with wealth changes, we must make use of the dynamic envelope first-order condition for second-best effort (Caputo, 2005), which is such that

$$\text{(A19)} \quad \mu = \frac{\int_{\underline{q}}^{\bar{q}} V f_e \, dq}{\psi''(e) - \int_{\underline{q}}^{\bar{q}} U f_{ee} \, dq}.$$

We do not attempt a full comparative statics analysis of this full equation. The procedure would be similar to the one just derived above, but it would involve a system of three nonlinear equations (i.e., the IR and IC' constraints plus the dynamic envelope condition in equation A19) in three dependent variables (λ, μ, and optimal effort e^*) and two independent variables (the wealth levels z_p and z_a).

Some final remarks on this dynamic envelope condition. The denominator is positive, since the first-order approach requires convexity of the agent's expected utility over effort (i.e. the second-order condition). This leaves us with the implication that $\text{sgn}[\mu] = \text{sgn}\left[\int_{\underline{q}}^{\bar{q}} V f_e dq\right]$, i.e., the sign of the multiplier on the IC' constraint is the same as the sign of the principal's marginal expected utility of agent effort, which in the presence of risk sharing we intuitively expect to be positive.

Intuitive Proof that $\frac{\partial w_q}{\partial z_p} = 0$ *when the Principal's Preferences Exhibit CARA*

Understanding why a principal whose preferences exhibit constant absolute risk aversion's wealth should not impact the optimal contract is equivalent to understanding why one can rescale the whole problem by a multiplicative constant and leave the optimal contract unchanged. The constant in this case is $\alpha = \exp\{-A_p z_p\}$.

The key insight is to recognize that, in addition to the objective function, the constraints can also be rescaled without changing any of the optimality conditions. Doing so leaves the constraint multipliers independent of principal wealth. That is, the constraints as written in equations 2 and 4 can be rewritten as

$$\text{(A20)} \quad \alpha\left\{\int_{\underline{q}}^{\bar{q}} U[z_a + w(q,\lambda,\mu)] f(q \mid e) dq - \psi(e)\right\} \geq \alpha\overline{U}$$

and

$$\text{(A21)} \quad \alpha\left\{\int_{\underline{q}}^{\bar{q}} U[z_a + w(q,\lambda,\mu)] f_e(q \mid e) dq - \psi'(e)\right\} = 0 .$$

But then, since the constant α multiplies every term of the Hamiltonian (i.e, the objective function and both constraints), one can divide the first-order condition (FOC) with respect to $w(q,\lambda,\mu)$ through by α, which implies that the optimal contract implicitly defined by the FOC does not depend on α. Likewise, canceling α from the constraints shows that λ and μ are independent of α. Finally, similar manipulations establish that the dynamic envelope condition for the second-best optimal effort level remains independent of α.

References

Caputo, M. 2005. *Foundations of Dynamic Economic Analysis*. Cambridge, UK: Cambridge University Press.